Vortices in Superfluid
MODD-Problems

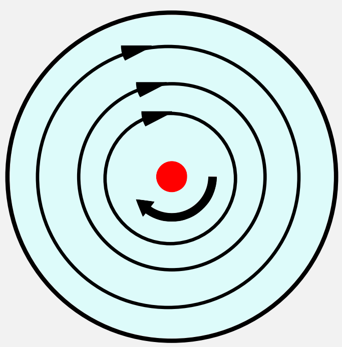
May 5, 2017

A. Steady filament (0.75)

Consider a cylindrical beaker (radius $R_0 \gg a$) of superfluid helium and a straight vertical vortex filament in its center Fig. 2.

A1 (0.25)

Plot the streamlines. Find out the velocity v at a point $\vec{r}$.



The streamlines are circular. From the circulation identity (1) it is obvious that $v = \kappa/r$.

- Streamlines are plotted correctly (one at least) 0.1
- $v = \frac{k}{r}$ 0.15

A2 (0.5)

Work out the free surface shape (height as a function of coordinate $z(\vec{r})$) around the vortex. Free fall acceleration is g. Surface tension can be neglected.

Consider a thin circular layer of the radius r. Equilibrium condition for its surface is given by the requirement

$$g\frac{dz}{dr} = \frac{v^2}{r} = \frac{\kappa^2}{r^3}. \quad (1)$$

This equation is satisfied by the surface profile

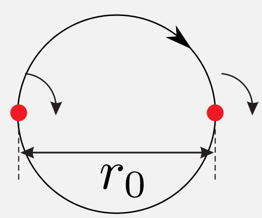
$$z(r) = [z_0] - \frac{\kappa^2}{2gr^2}. \quad (2)$$

- $\tan\alpha = \frac{k^2}{gr^3}$ or equivalent 0.25
- $z = [z_0] - \frac{k^2}{2gr^2}$ 0.25

B. Vortex motion (1.4)

B1 (0.25)

Consider two identical straight vortices initially placed at distance r_0 from each other as shown in Fig. 4. Find initial velocities of the vortices and draw their trajectories.

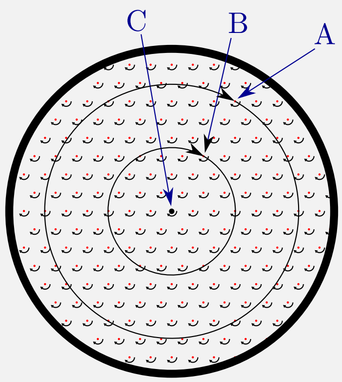


Being advected by each other's flow field, filaments will rotate around a point halfway between them. The velocity is given by $v_0 = \kappa/r_0$.

- Trajectories are plotted correctly 0.15
- Correct expression for velocity 0.1

B2 (0.15)

Draw the trajectories of vortices A, B, and C (located in the center).



- Trajectories are plotted correctly 0.15

B3 (0.4)

Find velocity $v(\vec{r})$ of a vortex positioned at $\vec{r}$.

Consider a circular path of radius $r \gg u$ around the beaker center. The circulation along this path is given by the number of vortices within it (vortex density per unit area is $(u^2\sqrt{3}/2)^{-1}$):

$$2\pi r v = 2\pi\kappa \frac{\pi r^2}{u^2\sqrt{3}/2}. \tag{3}$$

The velocity field

$$v = \frac{2\pi\kappa r}{u^2\sqrt{3}}. \tag{4}$$

- Expression for vortex density 0.2
- Correct expression for $v(r)$ 0.2

B4 (0.35)

Find the distance AB(t) between the vortices A and B at time t. Treat AB(0) as given.

This velocity pattern corresponds to the rotation of the lattice as a whole around the beaker center with angular velocity

$$\omega = \frac{2\pi\kappa}{u^2\sqrt{3}}. \tag{5}$$

$AB(t) = AB(0)$

- Correct answer 0.35

B5 (0.25)

Work out the "smoothed out" (omitting the lattice structure) free helium surface shape $z(\vec{r})$.

The surface shape is

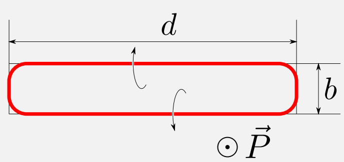
$$z(r) = [z_0] + \frac{\omega^2 r^2}{2g} = [z_0] + \frac{2\pi^2\kappa^2 r^2}{3gu^4}. \tag{6}$$

- Correct answer 0.25

C. Momentum and Energy (1.75)

C1 (0.3)

Consider a nearly rectangular vortex loop $b \times d$, $b \ll d$, Fig. 7. Indicate the direction of its momentum $\vec{P}$. Find out the momentum magnitude.



Momentum of a flat loop (see Introduction) is perpendicular to its plane and proportional to its area. For a rectangular loop the magnitude is $P = 2\pi\kappa\rho bd$.

- Correct direction of momentum 0.15
- Correct expression for momentum magnitude 0.15

C2 (0.7)

Calculate its energy U.

To produce equal magnetic and kinetic energy densities $B^2/(2\mu_0) = \rho v^2/2$, the magnetic field has to be $B = v\sqrt{\mu_0\rho} = \kappa\sqrt{\mu_0\rho}/r$. This field is generated by a current $I = 2\pi\kappa\sqrt{\rho/\mu_0}$. Energy of the wire loop can be found from the inductance $U = LI^2/2$. Inductance of a nearly rectangular wire loop:

$$L = \frac{\Phi}{I} = 2dI^{-1}\int_a^b \frac{\mu_0 I}{2\pi r}dr = \frac{\mu_0 d}{\pi}\log\frac{b}{a}. \tag{7}$$

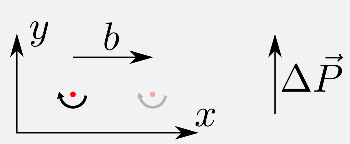
This gives for the energy

$$U = 2\pi\kappa^2\rho d\log\frac{b}{a} \tag{8}$$

- Integration limits are $\sim a$ and b 0.2
- Analogy with a magnitude field is used ($U = \frac{LI^2}{2}$, $L = \frac{\Phi}{I}$) or energy is calculated as $W = \int F\,dr$, where $F = \frac{dP}{dt}$ 0.2
- Correct expression for energy 0.3

C3 (0.75)

Suppose we shift a long straight vortex filament by a distance b in x direction, see Fig. 8. How much does the fluid momentum change? Indicate the momentum change direction. The filament length (constrained by the vessel walls) is d.



The momentum change is equal to the momentum of a long rectangular loop $P = 2\pi\kappa\rho bd$.

- The result of C1 used 0.3
- Momentum change is parallel to Y axis 0.1
- Correct direction of momentum change 0.15
- Correct expression for momentum change magnitude 0.2

Interestingly, this provides an alternative approach to find the energy of such a loop. Namely, if we slowly move one straight vortex in the velocity field of another, then we apply a force

$$F = 2\pi\kappa\rho d v = 2\pi\kappa\rho d \frac{\kappa}{r} = \frac{2\pi\kappa^2\rho d}{r}. \tag{9}$$

The work

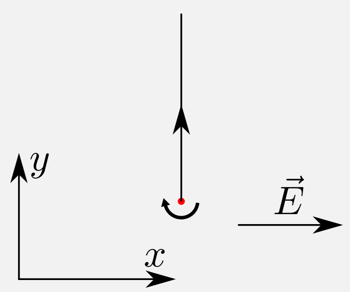
$$W = \int_a^b \frac{2\pi\kappa^2\rho d}{r} dr = 2\pi\kappa^2\rho d \log\frac{b}{a} \tag{10}$$

has to be performed to move it from distance a to b.

D. Trapped charges (2.85)

D1 (0.5)

Consider a straight vortex charged with uniform linear density $\lambda < 0$ in a uniform electric field $\vec{E}$. Draw the vortex trajectory. Find its velocity as a function of time.



Electric force $F = E\lambda d$ moves the vortex with velocity

$$v = \frac{F}{2\pi\kappa\rho d} = \frac{E\lambda}{2\pi\kappa\rho} \tag{11}$$

perpendicular to $\vec{E}$.

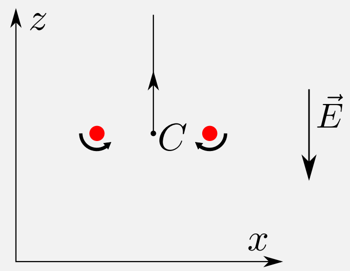
- Trajectory is straight line parallel to Y axis 0.1
- Correct direction of velocity 0.2
- Correct expression for velocity magnitude 0.2

D234

A circular vortex loop of radius R_0 initially charged with uniform linear density $\lambda < 0$ is placed in a uniform electric field $\vec{E}$ perpendicular to its plane, opposite to its momentum $\vec{P}_0$.

D2 (0.6)

Draw the trajectory of the loop center C. Find the radius of the loop as a function of time.



Electric force upon the loop $F = -2\pi E R_0 |\lambda|$ is constant and fluid momentum linearly depends on time

$$P = P_0 + 2\pi E R_0 |\lambda| t = 2\pi^2 \rho R^2 \kappa. \tag{12}$$

The loop is growing and its radius is increasing with time t

$$R = \sqrt{R_0^2 + \frac{E R_0 |\lambda| t}{\pi \rho \kappa}}. \tag{13}$$

- Trajectory is straight line along y 0.1
- Correct velocity direction 0.15
- $P(t)$ 0.15
- $2\pi^2 \rho R^2 \kappa$ 0.15
- Correct expression for $R(t)$ 0.05

D3 (1.5)

Find its velocity $v(t)$ as a function of time.

The loop velocity v can be easily found from a relationship between the energy change rate and the momentum change rate

$$\frac{dU}{dt} = Fv = \frac{dP}{dt}v. \tag{14}$$

This gives for the velocity

$$v = \frac{dU}{dP} \approx \frac{\kappa}{2R}\log\frac{R}{a} = \frac{\kappa \log\left(\sqrt{R_0^2 + ER_0|\lambda|t/(\pi\rho\kappa)}/a\right)}{2\sqrt{R_0^2 + ER_0|\lambda|t/(\pi\rho\kappa)}} \approx \frac{\kappa\log(R_0/a)}{2\sqrt{R_0^2 + ER_0|\lambda|t/(\pi\rho\kappa)}}. \tag{15}$$

This means that the vortex is moving in the direction of the force but its velocity is decreasing.

- Expression for $v \propto \frac{1}{R}\log(R)$ 1.0
- Correct expression for $v(t)$ 0.5

D4 (0.25)

The field is switched off at a time t^* when the velocity reaches the value $v^* = v(t^*)$. Find the loop velocity $v(t)$ at a later time $t > t^*$.

When $E = 0 \Rightarrow P = \text{const} \Rightarrow R = \text{const} \Rightarrow v = \text{const} \Rightarrow v(t) = v^*$.

- Correct expression for $v(t)$ 0.25

E. Influence of the boundaries (3.25)

Draw the trajectory of a straight vortex, initially placed at a distance h_0 from a flat wall. Find its velocity as a function of time.

E1 (0.5)

Well known technique of image charges (currents) in electrostatics (magnetostatics) can be directly used to solve this problem. Namely, the wall can be "substituted" with a reflected fictitious vortex on the other side of the wall. The velocity distribution of two vortices together in the upper semi-space is identical to the one produce by a single vortex above the wall. Indeed, the symmetry of the problem ensures that there is no flow through the plane of symmetry. Thus, a straight vortex line situated a distance h_0 above a flat wall with its image behave as a pair of vortices of opposite circulation a distance $2h_0$ apart. This means that the vortex moves along the wall with velocity

$$v = \frac{\kappa}{2h_0}. \tag{16}$$

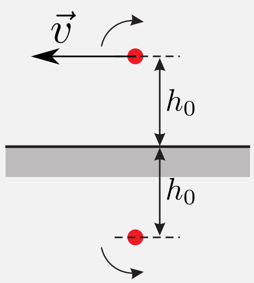


Illustration of the image method for the straight vortex filament near a flat wall

- Trajectory is plotted correctly 0.15
- Correct direction of velocity 0.1
- Correct expression for velocity magnitude 0.25

E234

Consider a straight vortex placed in a corner at a distance h_0 from both walls.

E2 (0.75)

What is the initial velocity v_0 of the vortex?

The velocity of the filament is given by superposition of the velocities $\vec{v}_1$, $\vec{v}_2$ and $\vec{v}_3$ induced by the image vortices 1, 2 and 3, respectively (see Fig. in E3 solution). One readily obtains

$$v_1 = \frac{\kappa}{2h_0}, \quad v_2 = \frac{\kappa}{2\sqrt{2}h_0}, \quad v_3 = \frac{\kappa}{2h_0}.$$

The modulus of the filament velocity at the initial moment is

$$v_0 = |\vec{v}_1 + \vec{v}_2 + \vec{v}_3| = \sqrt{2}v_1 - v_2 = \boxed{\frac{\kappa}{2\sqrt{2}h_0}}$$

- Ideas of using superposition principal and technique of image charges 0.25
- Correct expression for initial velocity magnitude 0.5

E3 (0.5)

Draw the trajectory of the vortex.

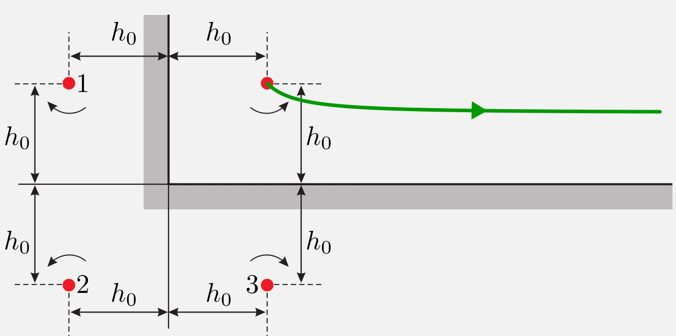


Image vortices in the corner.

- The trajectory has correct form 0.3
- Correct direction of initial velocity 0.2

E4 (1.5)

What is the velocity of the vortex v_∞ after very long time?

Energy for the system of vortices is proportional to

$$U_{\text{tot}} \propto \log \frac{\sqrt{x^2+y^2}}{a} - \log \frac{x}{a} - \log \frac{y}{a}. \tag{17}$$

The energy conservation implies that

$$C = \frac{x^2+y^2}{x^2y^2} = \frac{2}{h_0^2} \tag{18}$$

is constant along the trajectory. After very long time $y \to h_0/\sqrt{2}$ and the vortex velocity is

$$v_\infty = \frac{\kappa}{h_0\sqrt{2}}. \tag{19}$$

- $E = \text{const}$ 0.5
- Correct expression for velocity after very long time 1.5